Published by Institute of Physics Publishing for SISSA

Received: *August 3, 2005*
Revised: *November 22, 2005*
Accepted: *December 7, 2005*
Published: *January 9, 2006*

Non-BPS black holes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory coupled to gravity

Theodora Ioannidou

Mathematics Division, School of Technology, Aristotle University of Thessaloniki
Thessaloniki 54124, Greece
E-mail: ti3@auth.gr

Burkhard Kleihaus

Institut für Physik, Universität Oldenburg
Postfach 2503 D-26111 Oldenburg, Germany
E-mail: kleihaus@theorie.physik.uni-oldenburg.de

Abstract: We construct non-BPS regular and black hole solutions of $\mathcal{N} = 4$ SU(N) supersymmetric Yang-Mills theory coupled to Einstein gravity. Our numerical studies reveal a number of interesting phenomena when the gravitational constant $\alpha = M_{\rm YM}/gM_{\rm Planck}$ (where $M_{\rm Planck}$ is the Planck mass and $M_{\rm YM}$ is the monopole mass) is either weak (flat limit) or comparable to the Yang-Mills interaction. In fact, black hole solutions exist in a certain bounded domain in the $(\alpha, r_{\rm H})$ plane where $r_{\rm H}$ denotes the radius of the black hole horizon.

Keywords: Black Holes, Solitons Monopoles and Instantons, Supersymmetry and Duality.

Contents

1. Introduction

The $(3 + 1)$-dimensional $\mathcal{N} = 4$ SU(N) supersymmetric Yang-Mills theory broken to U$(1)^{N-1}$ can be studied as an effective field theory on N parallel D3-branes [1]. The existence of three strings junction connecting 3 parallel D3-branes was first conjectured in [2]. Subsequently, in [4] planar three strings junction have been found in explicit form. These states preserve 1/4 of the supersymmetries and correspond to static BPS spherically symmetric solutions of SU(3) Yang-Mills-Higgs theory. In [3], a class of static non-BPS dyon solutions of the aforementioned model has been derived which describes non-planar string junctions connecting N D3-branes. The solutions are spherical symmetric with electric charge determined by the Higgs vacuum expectations values (vevs).

In this paper, we extend this work by coupling the model with the Einstein gravity and construct the corresponding gravitating globally regular and black hole solutions. In the latter case, the strings meet at the horizon of the black hole and therefore, the string spectrum does not consists of junctions. Recall that, black holes that are solutions of supersymmetric theories can be associated with solitons [5].

From a complementary point of view one expects to obtain these BPS and non-BPS solutions from the supergravity equations of motion for any BPS and non-BPS state in the spectrum. The simplest BPS solutions of this kind are spherically symmetric black hole solutions of $\mathcal{N} = 2$ theories [6] and there existence strongly depends on the value of the charges and vacuum moduli [7]. Non-BPS composites could also exist [8]; however this is still an open question.

JHEP01(2006)032

2. The model

We consider a model in $(3+1)$ dimensions which consists of the Einstein-Hilbert action and the $\mathcal{N}=4$ SU(N) supersymmetric Yang-Mills action [9],

$$S = \int \left(\frac{1}{16\pi G} R + \mathrm{tr} \left\{ \kappa_1 F_{\mu\nu} F^{\mu\nu} + \kappa_2 \sum_{I=1}^{6} D_\mu \Phi^I D^\mu \Phi^I + \kappa_3 \sum_{I,J=1}^{6} [\Phi^I, \Phi^J]^2 \right\} \right) \sqrt{-g}\, d^4x. \tag{2.1}$$

Here Φ^I, $I=1,\ldots,6$ denote the six Higgs scalars, with covariant derivatives defined by: $D_\mu \Phi^I = \partial_\mu \Phi^I - i[A_\mu, \Phi^I]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, R is the scalar curvature, and g is the determinant of the metric. G denotes the gravitational coupling parameter and the constants κ_i are fixed as: $\kappa_1 = -\frac{1}{2}$, $\kappa_2 = -1$, $\kappa_3 = \frac{1}{2}$. In flat space this model can be considered as a dimensional reduction of the $(4+n)$-dimensional Yang-Mills theory, where n is the number of the extra dimensions and is (also) equal to the number of the Higgs fields of the $(3+1)$-dimensional $\mathcal{N}=4$ SU(N) supersymmetric Yang-Mills model.

For simplicity, we use the coordinates $r, z, \bar{z}$ on $\mathbb{R}^3$ where in terms of the usual spherical coordinates r, θ, ϕ the Riemann sphere variable z is given by $z = e^{i\phi}\tan(\theta/2)$. Then the Schwarzschild-like metric becomes

$$ds^2 = -A^2(r)B(r)\, dt^2 + \frac{1}{B(r)}\, dr^2 + \frac{4r^2}{(1+|z|^2)^2}\, dz d\bar{z}, \qquad B(r) = 1 - \frac{2m(r)}{r}, \tag{2.2}$$

where $A(r)$ and $B(r)$ are real functions and depend only on the radial coordinate r, and $m(r)$ is the mass function. The (dimensionfull) mass of the solution is $m_\infty \equiv m(\infty)$ and the square-root of the determinant is

$$\sqrt{-g} = iA(r)\, \frac{2r^2}{(1+|z|^2)^2}. \tag{2.3}$$

Also, the Einstein equations simplify to:

$$\frac{2}{r^2}\, m' = -8\pi G\, T_0^0, \qquad \frac{2}{r}\frac{A'}{A}\, B = -8\pi G\, \left(T_0^0 - T_r^r\right) \tag{2.4}$$

where prime denotes the derivative with respect to r and $T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_M - 2(\partial \mathcal{L}_M / g^{\mu\nu})$ is

$$\begin{aligned} T_{\mu\nu} = \mathrm{tr} \Big(& \kappa_1 (g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - 4 g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}) \Big) \\ & + \kappa_2 \sum_{I=1}^{6} \left(-2 D_\mu \Phi^I D_\nu \Phi^I + g_{\mu\nu} D_\alpha\, \Phi^I D^\alpha \Phi^I \right) + \kappa_3 g_{\mu\nu} \sum_{I,J=1}^{6} [\Phi^I, \Phi^J]^2 \; . \end{aligned} \tag{2.5}$$

The harmonic map ansatz to obtain SU(N) dyons [3] is of the form

$$\Phi^I = \sum_{j=0}^{N-2} \beta_j^I \left(P_j - \frac{1}{N} \right), \quad A_0 = \sum_{j=0}^{N-2} \delta_j \left(P_j - \frac{1}{N} \right), \quad A_z = i \sum_{j=0}^{N-2} \gamma_j [P_j, \partial_z P_j], \quad A_r = 0. \tag{2.6}$$

Here $\beta_j^I(r), \gamma_j(r), \delta_j(r)$ are real functions depending only on the radial coordinate r, and $P_j(z, \bar{z})$ are $N \times N$ hermitian projectors independent of r and orthogonal i.e. $P_i P_j = 0$ for $i \neq j$. Note that we are working in a real gauge, so that $A_{\bar{z}} = A_z^\dagger$.

The orthogonality of the projectors P_j means that the Higgs fields Φ^I are mutually commuting, i.e. $[\Phi^I, \Phi^J] = 0$, so they are simultaneously diagonalizable and this allows the eigenvalues to be interpreted as the positions of the strings in the transverse space.

The explicit form of the projectors is given as follows. Let f be the holomorphic vector

$$f = (f_0, \ldots, f_j, \ldots, f_{N-1})^t, \quad \text{where} \quad f_j = z^j \sqrt{\binom{N-1}{j}} \tag{2.7}$$

and $\binom{N-1}{j}$ denote the binomial coefficients. Define the operator Δ, acting on a vector $h \in \mathbb{C}^N$ as

$$\Delta h = \partial_z h - \frac{h\,(h^\dagger\,\partial_z h)}{|h|^2} \tag{2.8}$$

then P_j is defined as

$$P_j = \frac{(\Delta^j f)(\Delta^j f)^\dagger}{|\Delta^j f|^2}. \tag{2.9}$$

The particular form of these projectors corresponds to the requirement that the associated dyons are spherically symmetric (see [10] for more details).

It is convenient to make a change of variables to the following linear combinations

$$\beta_j^I = \sum_{k=j}^{N-2} b_k^I, \qquad c_j = 1 - \gamma_j - \gamma_{j+1}, \qquad \delta_j = \sum_{k=j}^{N-2} d_k, \qquad \text{for} \;\; j = 0, \ldots, N-2 \tag{2.10}$$

where we have defined $\gamma_{N-1} = 0$.

The magnetic charges, n_k, for $k = 1, \ldots, N-1$, can be read off from the large r behaviour of the magnetic field

$$\begin{aligned} B_i &= \frac{1}{2}\,\varepsilon_{ijk}\,F_{ij} \\ &\sim \frac{\widehat{x}_i}{2r^2} G \end{aligned} \tag{2.11}$$

where G is in the gauge orbit of

$$G_0 = \text{diag}(n_1,\, n_2 - n_1,\, \ldots,\, n_{N-1} - n_{N-2},\, -n_{N-1}). \tag{2.12}$$

In the case of maximal symmetry breaking, which we shall consider here, they are given by [3]

$$n_k = k(N-k), \qquad k = 1, \ldots, N-1. \tag{2.13}$$

Similarly, the large r asymptotic of the electric field

$$E_i \sim \frac{\widehat{x}_i}{2r^2} A_0 \tag{2.14}$$

allows the electric charges (which classically are real-valued) to be found from

$$A_0 = \sum_{j=0}^{N-2} 2\left(P_j - \frac{1}{N}\right)(r^2\delta_j{}')|_{r=\infty} \tag{2.15}$$

i.e. the electric charges are related to the $1/r$ coefficients of δ_j in a large r expansion.

After some algebra, it can be shown that substitution of (2.6) for $N = 3$ into the energy-momentum tensor leads to

$$\begin{aligned} T^0_0 - T^r_r &= \frac{8\kappa_1}{BA^2r^2}\left(c_0^2d_0^2 + c_1^2d_1^2\right) + \frac{8\kappa_1 B}{r^2}\left(c_0'^2 + c_1'^2\right) \\ &+ \frac{4\kappa_2 B}{3}\sum_I\left((b_0^I)'^2 + (b_1^I)'^2 + (b_0^I)'(b_1^I)'\right) \end{aligned} \tag{2.16}$$

$$\begin{aligned} T^0_0 &= \frac{4\kappa_1 B}{r^2}\left(c_0'^2 + c_1'^2\right) + \frac{4\kappa_1}{BA^2r^2}\left(c_0^2d_0^2 + c_1^2d_1^2\right) + \frac{2\kappa_2}{r^2}\sum_I\left(c_0^2(b_0^I)^2 + c_1^2(b_1^I)^2\right) \\ &+ \frac{4\kappa_1}{3A^2}\left(d_0'^2 + d_1'^2 + d_0'd_1'\right) + \frac{2\kappa_2 B}{3}\sum_I\left((b_0^I)'^2 + (b_1^I)'^2 + (b_0^I)'(b_1^I)'\right) \\ &+ \frac{4\kappa_1}{r^4}\left[(1-c_0^2)^2 + (1-c_1^2)^2 - (1-c_0^2)(1-c_1^2)\right], \end{aligned} \tag{2.17}$$

while variation of the energy density with respect to matter fields leads to the following system of ODE's

$$\begin{aligned} \frac{1}{A}\left(ABr^2\, b_0^{I'}\right)' &= 2\left(2c_0^2b_0^I - c_1^2b_1^I\right) \\ \frac{1}{A}\left(ABr^2\, b_1^{I'}\right)' &= 2\left(2c_1^2b_1^I - c_0^2b_0^I\right) \end{aligned} \tag{2.18}$$

$$\begin{aligned} BA\left(\frac{r^2}{A}\, d_0{}'\right)' &= 2\left(2c_0^2d_0 - c_1^2d_1\right) \\ BA\left(\frac{r^2}{A}\, d_1{}'\right)' &= 2\left(2c_1^2d_1 - c_0^2d_0\right) \end{aligned} \tag{2.19}$$

$$\begin{aligned} \frac{1}{A}\left(AB\, c_0'\right)' &= c_0\left[\frac{1}{r^2}(2c_0^2 - c_1^2 - 1) + \frac{1}{BA^2}\, d_0^2 + \frac{\kappa_2}{2\kappa_1}\sum_I (b_0^I)^2\right] \\ \frac{1}{A}\left(AB\, c_1'\right)' &= c_1\left[\frac{1}{r^2}(2c_1^2 - c_0^2 - 1) + \frac{1}{BA^2}\, d_1^2 + \frac{\kappa_2}{2\kappa_1}\sum_I (b_0^I)^2\right], \end{aligned} \tag{2.20}$$

with the appropriate boundary conditions due to the finiteness of the energy density.

Since we restrict to maximal symmetry breaking $\mathrm{SU}(3) \to \mathrm{U}(1)^2$, (i.e. to three D3-branes being distinct in transverse space), the boundary conditions on $c_j(r)$ are: $c_j(\infty) = 0$ where $j = 0, 1$. The remaining free parameters: $b_j^I(\infty)$, determine the vevs of the Higgs scalars since the ansatz (2.6) along the positive x_3-axis (that is, by setting $z = 0$) under the change of variables (2.10), results in

$$\Phi^I(r) = \frac{1}{3}\mathrm{diag}(2b_0^I + b_1^I, -b_0^I + b_1^I, -b_0^I - 2b_1^I) \tag{2.21}$$

from which the Higgs vevs can be read off in terms of $b_j^I(\infty)$.

By writing the components of (2.21) as

$$\Phi^I(r) = \mathrm{diag}(\Phi_1^I(r), \Phi_2^I(r), \Phi_3^I(r)) \tag{2.22}$$

the positions of the three D3-branes in the two-dimensional transverse space are given by

$$(x_\alpha^4, x_\alpha^5) = (\Phi_\alpha^1(\infty), \Phi_\alpha^2(\infty)) \quad \text{for} \quad \alpha = 1, 2, 3, \tag{2.23}$$

while their values for different r correspond to the positions of the strings which form the string junction and end on the D3-branes. (More details in section 3).

For globally regular solutions the boundary conditions of the matter profile functions are:

$$d_i(r=0) = 0, \quad c_i(r=0) = 1, \quad b_i^I(r=0) = 0, \quad m(r=0) = 0, \quad i = 0, 1, \tag{2.24}$$

and describe a string junction formed at the origin. However, black holes possess an event horizon at $r = r_{\mathrm{H}}$ determined by: $B(r_{\mathrm{H}}) = 0$, or (equivalently) by: $m(r_{\mathrm{H}}) = r_{\mathrm{H}}/2$. The horizon radius is a singular point of the differential equations and so, regularity of the solutions (due to (2.16)), impose the following boundary conditions:

$$d_i(r_{\mathrm{H}}) = 0, \quad i = 0, 1. \tag{2.25}$$

In what follows we consider only purely magnetic solutions i.e. $A_0 = 0$ which implies that $d_i(r) = 0$, for $i = 0, 1$. Then eqs. (2.18) and (2.20) evaluated at $r = r_{\mathrm{H}}$ yield

$$\begin{aligned}
\left[2\left(2c_0^2 b_0^I - c_1^2 b_1^I\right) - r^2 B' b_0^{I\prime}\right]_{\mathrm{H}} &= 0 \\
\left[2\left(2c_1^2 b_1^I - c_0^2 b_0^I\right) - r^2 B' b_1^{I\prime}\right]_{\mathrm{H}} &= 0
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
\left[c_0 \left\{\frac{1}{r^2}(2c_0^2 - c_1^2 - 1) + \frac{\kappa_2}{2\kappa_1}\sum_I (b_0^I)^2\right\} - B' c_0'\right]_{\mathrm{H}} &= 0 \\
\left[c_1 \left\{\frac{1}{r^2}(2c_1^2 - c_0^2 - 1) + \frac{\kappa_2}{2\kappa_1}\sum_I (b_1^I)^2\right\} - B' c_1'\right]_{\mathrm{H}} &= 0,
\end{aligned} \tag{2.27}$$

respectively.

3. Numerical simulations

Next we study the deformation of the classical soliton solutions of $\mathcal{N} = 4$ supersymmetric Yang-Mills equations to gravitating ones and black holes. In particular, we investigate the deformations as the gravitational parameter $\alpha = \sqrt{4\pi G}$ goes from zero to its maximum value. In this paper the asymptotic values of the Higgs field are fixed, i.e.

$$b_0^1 = -3/4, \quad b_1^1 = 1/4, \quad b_0^2 = b_1^2 = -1/3. \tag{3.1}$$

3.1 Globally regular solutions

Globally regular solutions describe self-gravitating monopoles. A heuristic argument [11, 12] suggests that they cannot persist if the coupling to gravity becomes too strong; i.e. for α of order one.

In particular, as α departs from zero a first branch of solutions emerges from the flat space monopole. When α reaches its maximal value: $\alpha_{\text{max}} \approx 1.349$ the first branch merges with a second one which bends back to a critical value $\alpha_{\text{cr}} \approx 1.345$. The mass of the second branch is slightly larger than of the first one, indicating an instability. On the second branch, when $\alpha \to \alpha_{\text{cr}}$ the minimum of $B(r)$ tends to zero and so a degenerate horizon is formed at: $r_{\text{deg}} = 2\alpha_{\text{cr}}$. In this limit the solution consists of an abelian part in the outside region $r_{\text{deg}} < r < \infty$, where the metric coincides with that of the extreme Reissner-Nordström black hole of charge two, the gauge functions $c_0(r)$ and $c_1(r)$ vanish identically and the Higgs functions $b_0^I(r)$ and $b_1^I(r)$ are constant; and of a non-abelian part in the inside region $0 \leq r < r_{\text{deg}}$, where the functions interpolate continuously between their values at the origin and at r_{deg}. As in [11, 12], the appearance of a double zero of B implies that the degenerate horizon is at an infinite physical distance from the origin, which means that the monopole is located inside the degenerate horizon.

These results are similar to the gravitating monopoles of SU(2) Einstein-Yang-Mills-Higgs theory which have been discussed extensively in [11, 12]. In figure 1 we plot the strings in the $x^4 - x^5$ plane for several values of α and observe that they do not change considerably as α increases from zero to one. Even for greater values of α the strings are deformed only slightly.

3.2 Black hole solutions

Next we study the construction of black hole solutions where three special cases arise. Case I describes embedded abelian solutions and occurs when $c_0(r) = c_1(r) = 0$; Case II describes embedded non-abelian solutions with gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$ and occurs when either $c_0(r) = 0$, $c_1(r) \neq 0$ (IIa) or $c_0(r) \neq 0$, $c_1(r) = 0$ (IIb); and Case III describes genuine non-abelian SU(3) solutions and occurs when both $c_0(r)$ and $c_1(r)$ are different from zero.

The numerical simulations show that: in case I and II the presence of abelian gauge fields impose a lower bound in the size of the black holes which can not be arbitrarily small; while in case II and III black holes cannot become arbitrarily large since (heuristically) their radius cannot exceed the radius of the non-abelian core of the monopole.

3.2.1 Case I

When $c_0(r) = c_1(r) = 0$ the Higgs profile functions are constant $b_0^I(r) = b_0^I(\infty)$, $b_1^I(r) = b_1^I(\infty)$ while the metric ones become

$$\begin{aligned} m(r) &= m_\infty - \frac{\alpha^2}{2}\frac{4}{r}, \\ A(r) &= 1. \end{aligned} \tag{3.2}$$

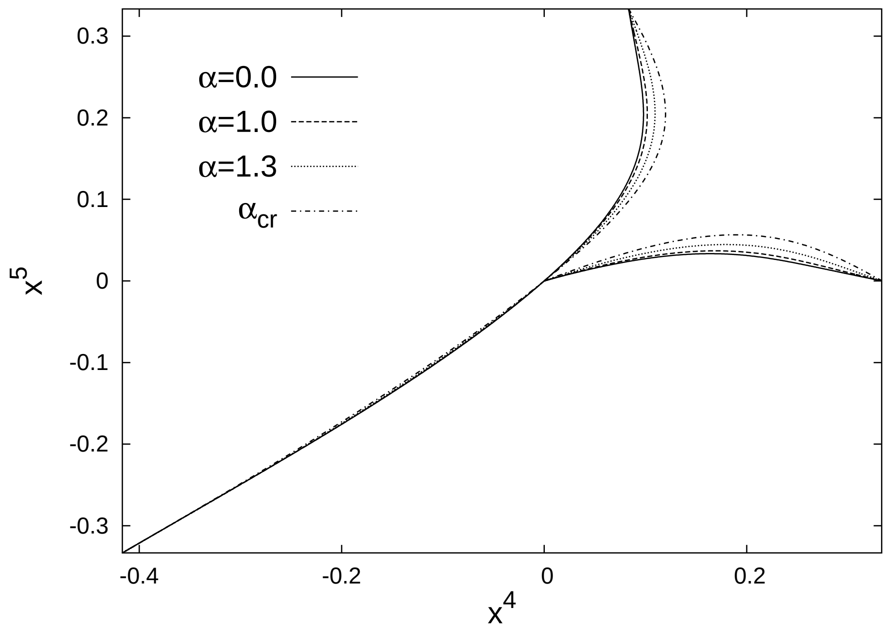

Figure 1: Three strings junction with boundary conditions given by (3.1) for several values of the gravitational coupling parameter.

where m_∞ is an integration constant. The solution corresponds to the Reissner-Nordström one with mass m_∞, charge two and event horizon given by: $r_{\rm H} = m_\infty + \sqrt{m_\infty^2 - 4\alpha^2}$. Note that, the black holes exist for $r_{\rm H} \geq 2\alpha$ and the equality holds for the extremal case. Figure 2 presents the domain of existence of the black holes in the $\alpha - r_{\rm H}$ plane and indicates that charge two Reissner-Nordström black holes exist for any α and $r_{\rm H}$ above the thick dashed line $r_{\rm H} = 2\alpha$.

3.2.2 Case II

Cases IIa and IIb are equivalent due to the symmetry of field equations under the interchange $(c_0, b_0^I) \leftrightarrow (c_1, b_1^I)$, (therefore, we concentrate on case IIa). When $c_0 = 0$ eqs. (2.18) are combined to a single one which after integration simplify to

$$2b_0^{I\prime} + b_1^{I\prime} = \frac{\rm const}{ABr^2}. \tag{3.3}$$

However, at the horizon $B(r_{\rm H}) = 0$ and thus for regular solutions, the constant has to vanish i.e.

$$b_0^I(r) = -\frac{1}{2} b_1^I(r) + \frac{1}{2}\left[2b_0^I(\infty) + b_1^I(\infty)\right]. \tag{3.4}$$

By setting $c_1(r) = w(r)/\sqrt{2}$ and $b_1^I(r) = h^I(r)$ the remaining differential equations reduce to

$$\left(ABr^2\, h^{I\prime}\right)' = 2w^2 h^I A \tag{3.5}$$

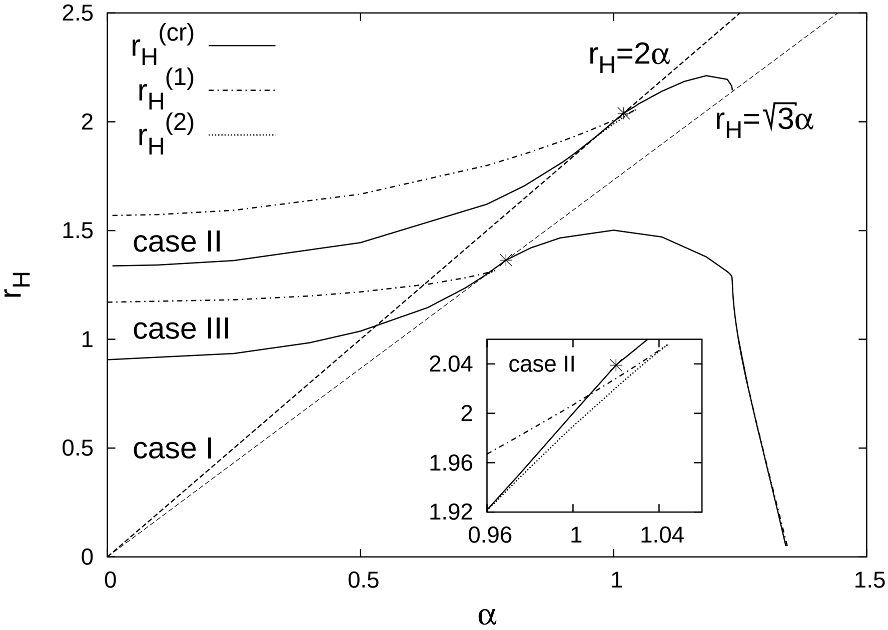

Figure 2: The domain of existence of black holes with boundary conditions given by (3.1).

$$\big(AB\,w'\big)' = w\left[\frac{1}{r^2}(w^2-1)+\sum_I (h^I)^2\right]A \tag{3.6}$$

$$m' = \alpha^2\left\{Bw'^2+\sum_I w^2(h^I)^2+\frac{B}{2}\sum_I r^2 {(h^I)'}^2+\frac{1}{2r^2}\left[(1-w^2)^2+3\right]\right\} \tag{3.7}$$

$$A' = \frac{2\alpha^2}{r}A\left\{w'^2+\frac{1}{2}\sum_I r^2{(h^I)'}^2\right\}, \tag{3.8}$$

which is the Einstein-Yang-Mills-Higgs system with gauge group SU(2) × U(1); multiple Higgs fields; and magnetic charge $Q=\sqrt{3}$ (due to the U(1) field).

Next, we argue why the black hole solutions of eqs. (3.5)–(3.8) exist only if $r_{\rm H}\geq\sqrt{3}\alpha$, as shown in figure 2: The constraint $B'(r_{\rm H})\geq 0$ is true since otherwise $B(r)$ would be negative near the horizon for $r>r_{\rm H}$ and equal to zero at some point r_0 (due to the asymptotic value $B(\infty)\to 1$). But such a point is a singular point of the equations of motion and therefore, a smooth solution is not guaranteed. Applying the above constraint in (3.7) gives the inequality

$$1-\frac{3\alpha^2}{r_{\rm H}^2}\geq 2\alpha^2\left\{\sum_I w_{\rm H}^2(h_{\rm H}^I)^2+\frac{1}{2r_{\rm H}^2}\left[(1-w_{\rm H}^2)^2\right]\right\}. \tag{3.9}$$

However, the right-hand side is non-negative implying that $r_{\rm H}\geq\sqrt{3}\alpha$ and therefore, no globally regular solutions exist in case II. *Remark:* note that in both cases I and II the lower bound of the horizon radius: $r_{\rm H}\geq Q\alpha$, is a consequence of the presence of an abelian field.

In addition, in case II an upper bound on black holes also exists. Figure 2 indicates that for fixed $\alpha \leq 0.959$ and varying $r_{\rm H}$ a first branch of solutions extends up to the maximal value $r_{\rm H}^{(1)}(\alpha)$, where it merges with a second branch which bends back to the critical value $r_{\rm H}^{(cr)}(\alpha)$ and finally bifurcates with the abelian Reissner-Nordström solution of case I. The curves $r_{\rm H}^{(1)}(\alpha)$ and $r_{\rm H}^{(cr)}(\alpha)$ are presented by the dash-dotted and solid lines, respectively, in figure 2.

Moreover, as shown in the inlet of figure 2, for $0.959 < \alpha < 1.044$ three branches of solutions exist. The first and second branch merge at $r_{\rm H}^{(1)}(\alpha)$, while the second and third branch merge at $r_{\rm H}^{(2)}(\alpha) < r_{\rm H}^{(1)}(\alpha)$ and the third one finally terminates at $r_{\rm H}^{(cr)}(\alpha)$. The curves $r_{\rm H}^{(1)}(\alpha)$ and $r_{\rm H}^{(2)}(\alpha)$ merge at $\alpha = 1.044$ while for larger values of α only one branch of solutions exists up to $r_{\rm H}^{(cr)}(\alpha)$. Note that, the upper $r_{\rm H}^{(cr)}(\alpha)$ and lower bound $r_{\rm H} = \sqrt{3}\alpha$ of the horizon radius coincide at some value of the gravitational parameter $\alpha = \alpha_{II}^{\max} \approx 1.235$ above which no case II black hole solutions exist.

At the critical radius $r_{\rm H}^{(cr)}(\alpha)$ a bifurcation with the Reissner-Nordström solution of case I occurs only when $\alpha \leq \alpha_{II}^{*} = 1.02$. For $\alpha > \alpha_{II}^{*}$, the branches terminate since a degenerate horizon is formed. In the limit $r_{\rm H} \to r_{\rm H}^{(cr)}(\alpha)$ the local minimum of the metric function B becomes equal to zero at $r_{\rm deg} = 2\alpha$. The limiting solution can be described as follows: for $r > r_{\rm deg}$ the metric and gauge potential correspond to an extremal abelian black hole with magnetic charge two and constant Higgs field. However, inside the degenerate horizon $r_{\rm H} \leq r < r_{\rm deg}$ a non-abelian core persists. The formation of the degenerate horizon is similar to the one of the globally regular solutions, discussed in section 3.1.

In order to understand the transition to abelian black holes on a qualitative level, one can consider the monopole as an extended object with a black hole inside the non-abelian core. When the black hole becomes larger than the monopole core, the non-abelian fields outside the horizon can not persist any more and transform to abelian ones. The presence of different branches of solutions indicates the existence of instability, i.e. solutions with the largest masses are less stable. Figure 3 demonstrates the bifurcation with the abelian solution of case I and presents the mass of the solutions in all cases for $\alpha = 0.5$.

The functions $B(r)$, $A(r)$, $f(r)$, $h^1(r)$ and $h^2(r)$ are plotted in figures 4a–4d for different values of the horizon radius when $\alpha = \sqrt{1.3}$. These functions are plotted in the region $r \in [r_{\rm H}, 3]$ so that the formation of the horizon is demonstrated.

3.2.3 Case III

For the genuine SU(3) black holes no abelian gauge field is present and therefore, no lower bound on the horizon radius exists. Indeed, for vanishing horizon radius the black hole solutions tend pointwise to the globally regular solutions. However, as in case II, the solutions bifurcate with non-extremal black holes when α is small; while a degenerate horizon is formed for large values of α with transition point at: $\alpha_{III}^{*} \approx 0.787$.

Let us consider first the case $\alpha < \alpha_{III}^{*}$. For α small, a first branch of black hole solutions emerges from the globally regular solutions with increasing horizon radius which merges with a second branch at the maximal value $r_{\rm H}^{(1)}(\alpha)$, and then bends back to a critical value $r_{\rm H}^{(cr)}(\alpha)$. At this critical value the second branch bifurcates with the first branch of

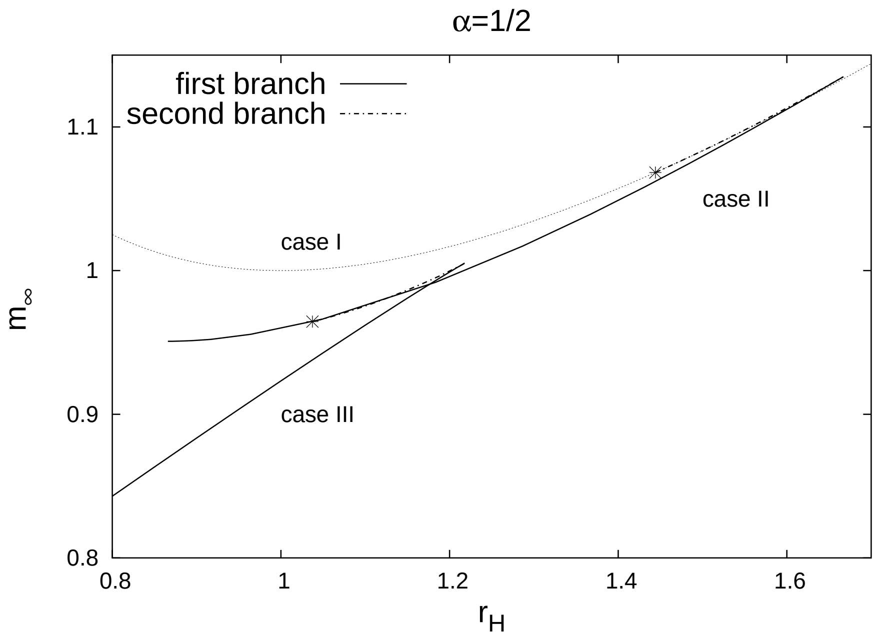

Figure 3: The mass of the black hole solutions of case I, II and III as function of $r_{\rm H}$ for $\alpha = 0.5$. (Asterisk indicates the bifurcation point.)

case II (in contrast with the abelian solutions of case II), as demonstrated in figure 3. Similarly with case II, three branches of solutions exist for $0.74 \le \alpha \le 0.766$ and (only) one for $0.766 \le \alpha \le \alpha^*_{III}$; which bifurcate with the case II ones at the limit: $r_{\rm H} \to r_{\rm H}^{(cr)}(\alpha)$.

The transition to the $SU(2) \times U(1)$ black hole solutions of case II can be explained by the following argument: the Higgs fields at infinity define two vector boson masses and core radii $m_0^2 = \left[(b_0^1)^2 + (b_0^2)^2\right]_\infty$, $R_0 = 1/m_0$ and $m_1^2 = \left[(b_1^1)^2 + (b_1^2)^2\right]_\infty$, $R_1 = 1/m_1$, leading to the exponential decay of c_0 and c_1, respectively. Since $m_0 > m_1$ (due to (3.1)), c_0 is essentially zero outside R_0 where $SU(3)$ breaks to $SU(2) \times U(1)$. Consequently, if the horizon radius is of the order of R_0, an $SU(2)$ gauge field and a $U(1)$ field exists outside the horizon.

For $\alpha > \alpha^*_{III}$ two distinguished cases exist when either $\alpha \le \alpha_{II}^{\rm max}$ or $\alpha \ge \alpha_{II}^{\rm max}$. In the first case, only one branch of solutions exists which terminates at $r_{\rm H} = r_{\rm H}^{(cr)}(\alpha)$ and a degenerate horizon forms at $r_{\rm deg} = \sqrt{3}\alpha$. The limiting solution coincides with the extremal case II solution in the outside region $r > \sqrt{3}\alpha$ when: $c_0(r) = 0$, $c_1(r) = w(r)$, $2b_0^I(r) + b_1^I(r) = \text{const}$, $b_1^I(r) = h^I(r)$ and $B(r) = A(r)$. In the inside region $r < \sqrt{3}\alpha$, the functions $c_0(r)$ and $2b_0^I(r) + b_1^I(r)$ are non-trivial; and the situation is similar to the solutions of case II when the extremal abelian solution is replaced by the extremal case II solution. Thus, we see again that a horizon is formed at $r_{\rm deg}$ when the monopole core becomes too massive. However, for the genuine $SU(3)$ solutions there is (still) a non-abelian gauge field outside $r_{\rm deg}$ due to the lighter mass m_1.

In the second case the extremal case II solutions do not exist since $\alpha > \alpha_{II}^{\rm max}$. In fact as the horizon radius tends to $r_{\rm H}^{(cr)}(\alpha)$, a degenerate horizon forms at $r_{\rm deg} = 2\alpha$. In this

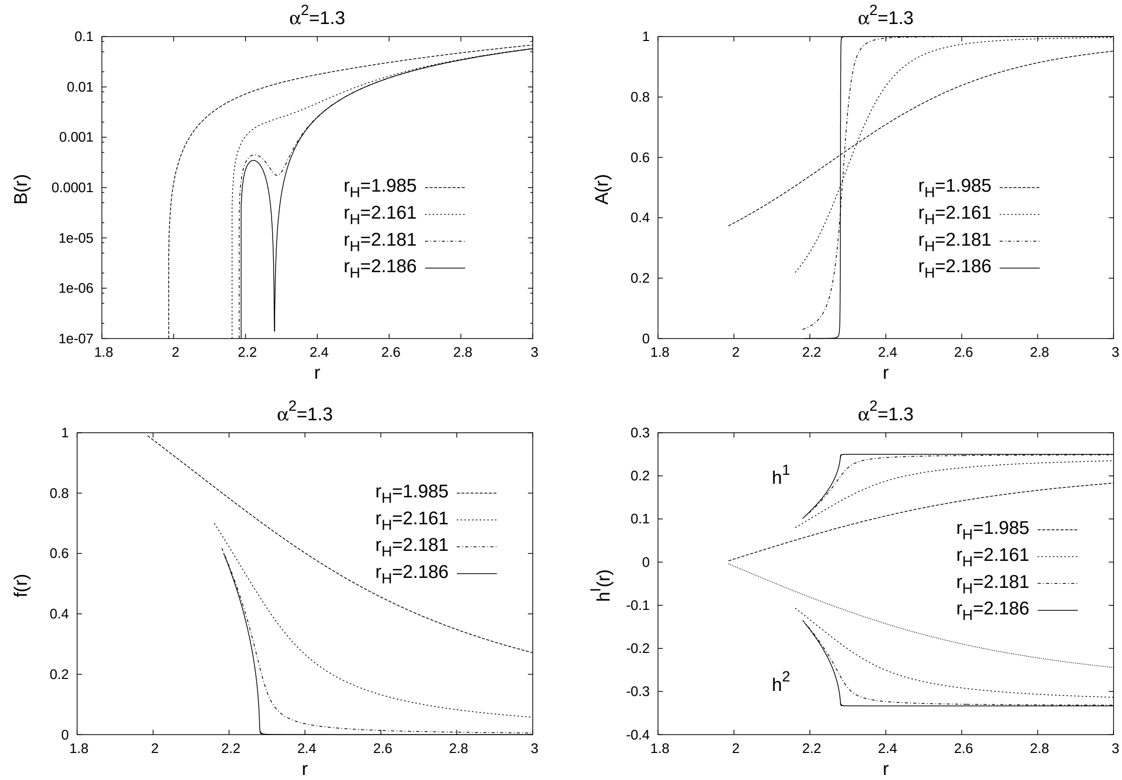

Figure 4: The metric functions $B(r)$ (upper left), $A(r)$ (upper right), the gauge field function $f(r)$ (lower left) and the Higgs field functions $h^1(r)$, $h^2(r)$ (lower right) in terms of r_H and $\alpha = \sqrt{1.3}$.

limit the solution in the outside region is abelian with magnetic charge two; while in the inside region is non-abelian (as case II). For larger values of α, two branches of solutions exist which m erge at some $r_\mathrm{H}^{(1)}(\alpha)$; while a degenerate horizon forms on the second branch as $r_\mathrm{H} \to r_\mathrm{H}^{(cr)}(\alpha)$. There are no genuine non-abelian black hole solutions for $\alpha > \alpha_\mathrm{max}$.

4. String junction

Recall that the vevs of the Higgs fields determine the position of the strings in transverse space due to (2.23). For globally regular solutions the Higgs functions vanish at the origin and the three strings are joint in transverse space. However, in black holes the origin is replaced by their horizon and thus the strings are not connected in general.

Next we discuss the string interpretation of our solutions in all three cases:

Case I is the simplest one since the symmetry breaking is $\mathrm{U}(1)^2$. The Higgs field equals its vacuum value everywhere in space and so the strings degenerate to a point on the brane.

In case II, the symmetry breaking is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and thus the Higgs functions are linearly dependent as shown in (3.4). So the first string degenerates to the point

$$(x_1^4, x_1^5) = \left(\frac{1}{3}\left(2b_0^1(\infty) + b_1^1(\infty)\right), \frac{1}{3}\left(2b_0^2(\infty) + b_1^2(\infty)\right)\right) \tag{4.1}$$

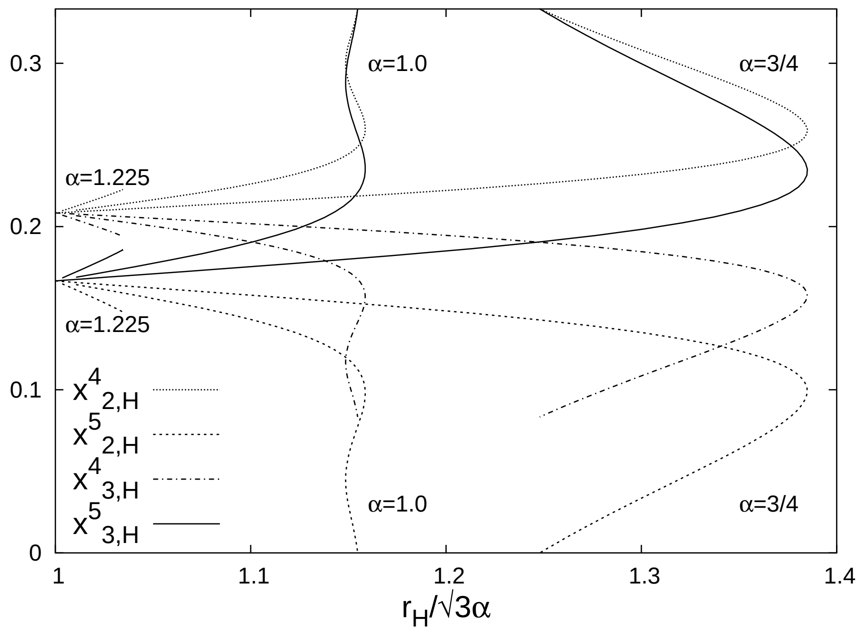

Figure 5: The coordinates of the strings 2 and 3 at the horizon as function of the scaled horizon radius $r_{\rm H}/\sqrt{3}\alpha$ for $\alpha = 1.225, 1, 3/4$.

since one of the branes decouples when the non-abelian symmetry group is reduced from SU(3) to SU(2). Recall that in case IIa, the Higgs profile functions are related since

$$h^1(r)/h^1(\infty) = h^2(r)/h^2(\infty) \tag{4.2}$$

which implies that x^5_a is a linear function of x^4_a, for $a = 2, 3$. Therefore, the strings 2 and 3 form straight lines in transverse space connecting the branes at $\big(x^4_a(\infty), x^5_a(\infty)\big)$ to the point $\big(x^4_a(r_{\rm H}), x^5_a(r_{\rm H})\big)$.

Figure 5 presents the string coordinates at the horizon $(x^4_i, x^5_i)|_{r_{\rm H}}$ (for $i = 1, 2$) as functions of $r_{\rm H}$ when $\alpha = 1.225, 1, 0.75$ and shows that the strings are not joint in the non-extremal case since $\left(x^4_2, x^5_2\right)|_{r_{\rm H}} \neq \left(x^4_3, x^5_3\right)|_{r_{\rm H}}$ for $r_{\rm H} > \sqrt{3}\alpha$. However, in the extremal case $r_{\rm H} = \sqrt{3}\alpha$ the strings 2 and 3 are jointed at the horizon since $b^I_1(r_{\rm H}) = 0$.

Next we investigate the consequences of the transition to abelian black holes when $\alpha \leq \alpha^*_{II}$ and the formation of a degenerate horizon when $\alpha > \alpha^*_{II}$. Figure 5 shows that for $\alpha \leq \alpha^*_{II}$ the coordinates $x^4_{2,\rm H}$, $x^5_{3,\rm H}$ increase monotonously with $r_{\rm H}$ up to a maximal value where the transition to abelian black holes occurs. (Similarly, $x^5_{2,\rm H}$, $x^4_{3,\rm H}$ decrease monotonously). At the transition point the coordinates $(x^4_{a,\rm H}, x^5_{a,\rm H})$ are equal to $(x^4_a(\infty), x^5_a(\infty))$ — hence the strings degenerate to points on the branes. Non-degenerate strings cannot exist for black holes larger than the non-abelian core. For $\alpha > \alpha^*_{II}$ a degenerate horizon forms outside the non-abelian core when $r_{\rm H} \to r^{(cr)}_{\rm H}$ and the strings stretch between the (event) horizon and the degenerate one.

In case III, the strings do not degenerated at a point as shown in figure 6 where the $x^4 - x^5$ plane for $r_H = 0.1, 1$ and $\alpha = 1.0$ is plotted. However, the separation of the end

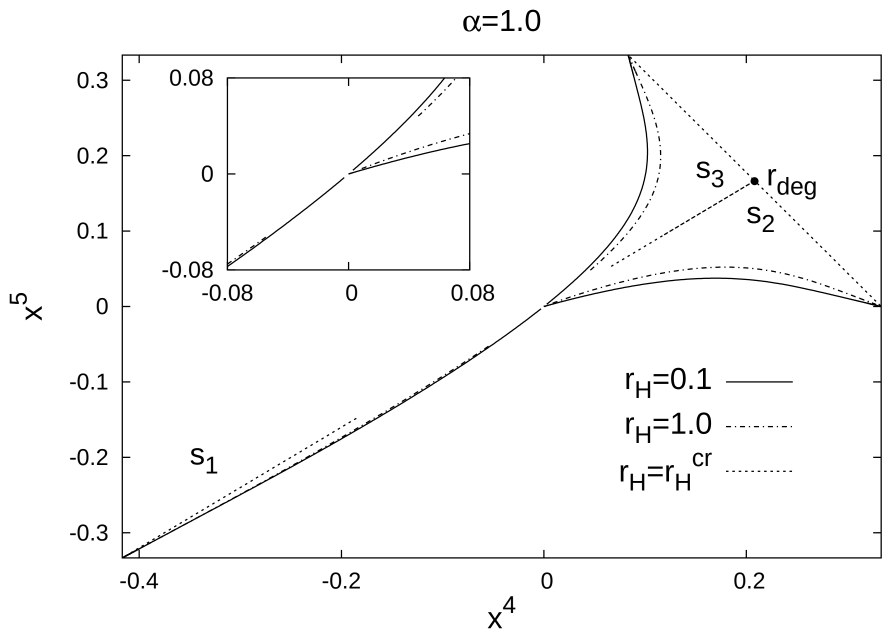

Figure 6: Case III three strings in the $x^4 - x^5$ plane for $r_{\mathrm{H}} = 0.1, 1.0, r_{\mathrm{H}}^{(cr)}$ and $\alpha = 1$.

points at the horizon gets smaller for small horizon radius, and the strings joint as the black hole solutions approach the globally regular solutions (in the limit of a vanishing horizon radius). Similarly to case II, for $\alpha \leq \alpha^*_{III}$ the SU(3) black holes bifurcate with non-extremal SU(2) $\times$ U(1) black holes which implies that one string degenerates to a point, while the others are associated with the gauge group SU(2). On the other hand, for $\alpha^*_{III} < \alpha \leq \alpha^{\max}_{II}$, a degenerate horizon forms at r_{deg} leaving three non-degenerate strings in the inside and two strings in the outside region as demonstrated in figure 6. (Note that, the existence of the string in the outside region is due to the presence of the non-abelian SU(2) gauge fields). In fact, since (in the outside region) the solutions form extremal SU(2)$\times$U(1) black holes the two non-degenerate strings s_2 and s_3 are joint at r_{deg} and form a straight line, whereas string s_1 degenerates to a point. Finally, for $\alpha > \alpha^{\max}_{II}$ there are no non-abelian gauge fields outside the degenerate horizon and so, the strings (in the outside regions) are degenerated to points. In contrast, in the inside region three non-degenerated strings persists.

5. Extreme SU(4) black hole solutions

In section 3.2.2 it was shown that a special case occurs when one of the gauge functions becomes zero leading to SU(2) $\times$ U(1) solutions with an additional magnetic charge. Here we extend this work in the SU(4) case to construct SU(3) $\times$ U(1) solutions.

The energy-momentum tensor, for purely magnetic gauge potential, is

$$T_0^0 - T_r^r = \frac{4\kappa_1 B}{r^2}\left(3c_0'^2 + 4c_1'^2 + 3c_2'^2\right)$$

$$+2\kappa_2 B\sum_I\left(\frac{3}{4}(b_0^I)'^2+(b_1^I)'^2+\frac{3}{4}(b_2^I)'^2+(b_0^I)'(b_1^I)'+\frac{1}{2}(b_0^I)'(b_2^I)'+(b_1^I)'(b_2^I)'\right)$$

$$T_0^0 = \frac{2\kappa_1 B}{r^2}\left(3c_0'^2+4c_1'^2+3c_2'^2\right)+\frac{\kappa_2}{r^2}\sum_I\left(3c_0^2(b_0^I)^2+4c_1^2(b_1^I)^2+3c_2^2(b_2^I)^2\right)$$

$$+\kappa_2 B\sum_I\left(\frac{3}{4}(b_0^I)'^2+(b_1^I)'^2+\frac{3}{4}(b_2^I)'^2+(b_0^I)'(b_1^I)'+\frac{1}{2}(b_0^I)'(b_2^I)'+(b_1^I)'(b_2^I)'\right)$$

$$+\frac{2\kappa_1}{r^4}\left(\frac{9}{2}c_0^4+8c_1^4+\frac{9}{2}c_2^4-3c_0^2-4c_1^2-3c_2^2-6c_0^2c_1^2-6c_2^2c_1^2+5\right), \tag{5.1}$$

while the equations of motion for the matter profile functions are

$$\begin{aligned}
\frac{1}{A}\left(ABr^2\, b_0^{I'}\right)' &= 6c_0^2 b_0^I - 4c_1^2 b_1^I\\
\frac{1}{A}\left(ABr^2\, b_1^{I'}\right)' &= 8c_1^2 b_1^I - 3c_0^2 b_0^I - 3c_2^2 b_2^I\\
\frac{1}{A}\left(ABr^2\, b_2^{I'}\right)' &= 6c_2^2 b_2^I - 4c_1^2 b_1^I\\
\frac{1}{A}\left(AB\, c_0'\right)' &= c_0\left[\frac{1}{r^2}(3c_0^2-2c_1^2-1)+\frac{\kappa_2}{2\kappa_1}\sum_I(b_0^I)^2\right]\\
\frac{1}{A}\left(AB\, c_1'\right)' &= c_1\left[\frac{1}{r^2}(4c_1^2-\frac{3}{2}c_0^2-\frac{3}{2}c_2^2-1)+\frac{\kappa_2}{2\kappa_1}\sum_I(b_1^I)^2\right]\\
\frac{1}{A}\left(AB\, c_2'\right)' &= c_2\left[\frac{1}{r^2}(3c_2^2-2c_1^2-1)+\frac{\kappa_2}{2\kappa_1}\sum_I(b_2^I)^2\right].
\end{aligned} \tag{5.2}$$

Note that, setting $c_0(r)=0$, the Higgs field equations imply

$$b_0^I(r) = \text{const} - \frac{2}{3}b_1^I(r) - \frac{1}{3}b_2^I(r), \tag{5.3}$$

and scaling the gauge field functions as

$$c_1(r) = w_0(r)/\sqrt{2}, \qquad c_2(r) = w_1(r)\sqrt{2/3}\,, \tag{5.4}$$

the SU(4) equations transform to the SU(3) ones when c_a, b_a^I are replaced by w_a, h_a^I, respectively. However, there is the extra term $-6/(2r^4)$ in T_0^0 due to the U(1) field with charge $\sqrt{6}$. Consequently, solutions do exist only if $r_{\rm H} \geq \sqrt{6}\alpha$.

In the extreme case (i.e. for $r_{\rm H} = \sqrt{6}\alpha$) the boundary conditions for the gauge and Higgs functions at the horizon are

$$w_0(r_{\rm H}) = w_1(r_{\rm H}) = 1, \qquad h_0^I(r_{\rm H}) = h_1^I(r_{\rm H}) = 0 \tag{5.5}$$

while the string coordinates in transverse space are

$$\begin{aligned}
(x_1^4(r)\,,\ x_1^5(r)) &= \vec{C}\\
(x_2^4(r)\,,\ x_2^5(r)) &= -\frac{1}{3}\vec{C}+\frac{1}{3}\left(2h_0^1(r)+h_1^1(r)\,,\ 2h_0^1(r)+h_1^1(r)\right)\\
(x_3^4(r)\,,\ x_3^5(r)) &= -\frac{1}{3}\vec{C}+\frac{1}{3}\left(-h_0^1(r)+h_1^1(r)\,,\ -h_0^1(r)+h_1^1(r)\right)\\
(x_4^4(r)\,,\ x_4^5(r)) &= -\frac{1}{3}\vec{C}+\frac{1}{3}\left(-h_0^1(r)+2h_1^1(r)\,,\ -h_0^1(r)+2h_1^1(r)\right).
\end{aligned} \tag{5.6}$$

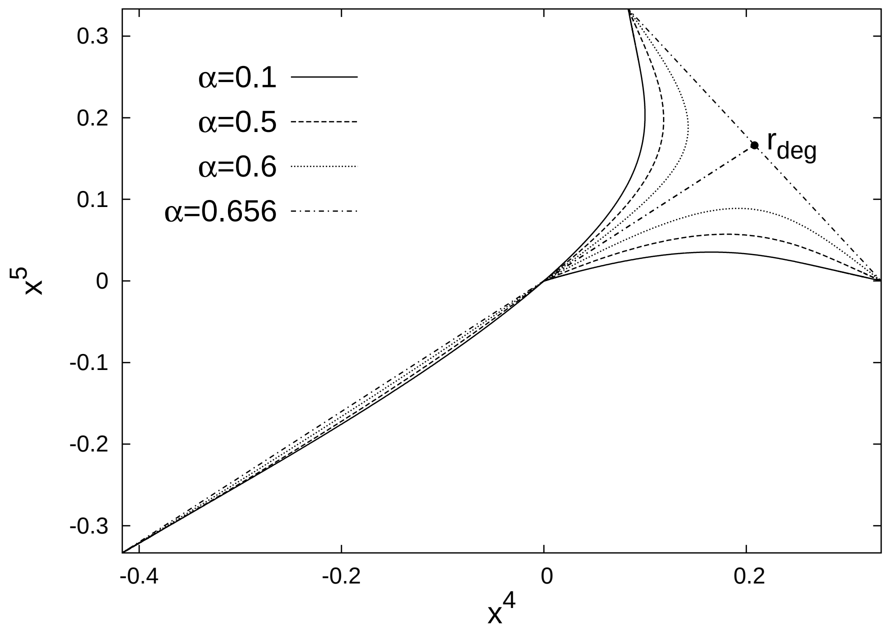

Figure 7: The three strings of the extremal $\mathrm{SU}(3) \times \mathrm{U}(1)$ solutions in the $x^4 - x^5$ plane for $\alpha = 0.1, 0.5, 4, 0.6$ and $\alpha_{\max} = 0.656$.

Here $\vec{C} = \text{const}$ which implies that the first string degenerates to a point.

We computed the extreme solutions for a wide range of values of α, except the limit $\alpha \to 0$. In this region, only one branch has been found which terminates at the maximal value $\alpha_{\max} \approx 0.656$; while at the limit $\alpha \to \alpha_{\max}$ the formation of a (second) degenerate horizon at $r_{\text{deg}} = 3\alpha$ has been observed. In the outside region, $r_{\text{deg}} < r < \infty$, the limiting solution corresponds to an extremal $\mathrm{SU}(2) \times \mathrm{U}(1)$ black hole and the $\mathrm{U}(1)$ field has charge $Q = 3$ since $2h_0^I(r) + h_1^I(r)$ is constant; $w_0(r)$ vanishes identically; whereas $w_1(r)$ is non-trivial. In the inside region, $r_{\mathrm{H}} \leq r < r_{\text{deg}}$, the functions of the limiting solutions interpolate continuously between their values at r_{H} and r_{deg}. A degenerate horizon is formed since by increasing the coupling to gravity the gravitational radius becomes of the order of the core of the heaviest gauge field component. But the gauge field components with smaller mass exist outside the degenerate horizon and that is why the transition to an extremal $\mathrm{SU}(2) \times \mathrm{U}(1)$ black hole in the outside region occurs.

The formation of the degenerate horizon do affect the strings as shown in figure 7 which presents the three strings in the $x^4 - x^5$ plane for several values of α when $\vec{C} = 0$. Note that, for α small, the strings are similar to those of the globally regular solutions; however as α increases the strings deform and in the limit $\alpha \to \alpha_{\max}$ the picture changes drastically. For $r_{\text{deg}} < r < \infty$ the second string consists of a single point $(x_2^4(\infty), x_2^5(\infty))$ which tends to the origin as r decreases from $r_{\text{deg}} \to r_{\mathrm{H}}$. The third and fourth string form straight lines for $r_{\text{deg}} < r < \infty$ and finally, merge when $r \to r_{\text{deg}}$. In the limit, $r \to r_{\mathrm{H}}$ these two strings merge and form a straight line passing through origin.

6. Conclusion

String theory assumes that space-time possess more than four dimensions and that these dimensions are compactified on a scale of the Planck length. The extra dimensions offer a solution to the hierarchy problem and assume that all known interactions (except gravity) are confined on a three-dimensional brane of a $(4+n)$-dimensional spacetime. Therefore, it would be of interest to construct our string and black hole solutions from the corresponding Yang-Mills-Einstein equations defined in higher dimensions. It has been shown in [13], that an $(4+n)$ dimensional Yang-Mills-Einstein model exists with n Higgs fields and n dilatons [13] as long as the metric and matter fields are independent of the extra coordinates. Thus, it would be interesting to construct our solutions by solving the $(4+2)$-dimensional Yang-Mills-Einstein equations.

Also, an open and more difficult task would be to derive the same solutions by solving the corresponding $\mathcal{N}=4$ supergravity equations of motion.

Acknowledgments

The authors acknowledge valuable discussion with David Tong and Tassos Petkou. B.K. gratefully acknowledges support by the DFG.

References

[1] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[2] J.H. Schwarz, *Lectures on superstring and M-theory dualities*, *Nucl. Phys.* **55B** *(Proc. Suppl.)* (1997) 1 [hep-th/9607201].

[3] T.A. Ioannidou and P.M. Sutcliffe, *Non-BPS string junctions and dyons in $N=4$ super–Yang-Mills*, *Phys. Lett.* **B 467** (1999) 54 [hep-th/9907157].

[4] K. Hashimoto, H. Hata and N. Sasakura, *3-string junction and BPS saturated solutions in* SU(3) *supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 431** (1998) 303 [hep-th/9803127].

[5] G.W. Gibbons, *Supersymmetry, supergravity and related topics*, F. Aguila, J.A. Azcarraga, L.E. Ibáñez eds., World Scientific, Singapore 1985.

[6] S. Ferrara, R. Kallosh and A. Strominger, *$N=2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[7] G. Moore, *Arithmetic and attractors*; hep-th/9807087; *Attractors and arithmetic*, hep-th/9807056.

[8] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049].

[9] As E. Radu (private communication) pointed out, coupling of supersymmetric Yang-Mills to gravity before descending to (3+1) dimensions leads to a different effective action.

[10] W.J. Zakrzewski, *Low dimensional sigma models*, IOP, Bristol 1989.

[11] K. Lee, V.P. Nair and E.J. Weinberg, *Black holes in magnetic monopoles*, *Phys. Rev.* **D 45** (1992) 2751 [hep-th/9112008].

[12] P. Breitenlohner, P. Forgacs and D. Maison, *Gravitating monopole solutions*, *Nucl. Phys.* **B 383** (1992) 357; *Gravitating monopole solutions, 2*, *Nucl. Phys.* **B 442** (1995) 126 [gr-qc/9412039].

[13] Y. Brihaye, F. Clement and B. Hartmann, *Static solutions of a 6-dimensional Einstein-Yang-Mills model*, hep-th/0406034.